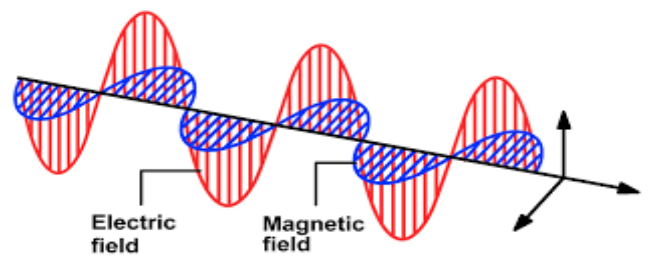
KINGDOM OF SAUDI ARABIA
AL-IMAM MUHAMMAD BIN SAUD
ISLAMIC UNIVERSITY
FACULTY OF SCIENCES
DEPARTMENT OF PHYSICS

المملكة العربية السعودية
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم
قسم الفيزياء

Electromagnetic Field (PHY 321)



Dr. Naglaa Abdel All

1445 / 2024

Chapter1 : Vector Analysis

Reference book:

Sadiku M., ***Elements of Electromagnetic***, 2nd Edition, Saunders College (1995).

1- Vector Multiplication

ضرب المتجهات

1-1- Scalar Product (or dot product)

Consider $\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$ and $\vec{B} = B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z$. We define the scalar product (or dot product) of vectors $\vec{A}$ and $\vec{B}$ as

كاجية

$$\boxed{\vec{A}\cdot\vec{B} \equiv |\vec{A}||\vec{B}|\cos\theta}$$ where θ is the smaller angle between $\vec{A}$ and $\vec{B}$

We can prove that $\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$

$$\vec{a}_x\cdot\vec{a}_x = \vec{a}_y\cdot\vec{a}_y = \vec{a}_z\cdot\vec{a}_z = 1$$

$$\vec{a}_x\cdot\vec{a}_y = \vec{a}_x\cdot\vec{a}_z = \vec{a}_y\cdot\vec{a}_z = 0$$

$$\vec{a}_x\times\vec{a}_x = \vec{a}_y\times\vec{a}_y = \vec{a}_z\times\vec{a}_z = 0$$

$$\vec{a}_x\times\vec{a}_y = \vec{a}_z$$

$$\vec{a}_y\times\vec{a}_z = \vec{a}_x$$

$$\vec{a}_z\times\vec{a}_x = \vec{a}_y$$

1-2- Vector Product (or Cross Product)

We define the vector product of vectors $\vec{A}$ and $\vec{B}$ as

$$\boxed{\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\vec{a}_n}$$

where $\vec{a}_n$ is a unit vector (i.e., $|\vec{a}_n| = 1$) normal to the plane containing $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\vec{a}_x - (A_xB_z - A_zB_x)\vec{a}_y + (A_xB_y - A_yB_x)\vec{a}_z$$

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z \qquad (A_x, A_y, A_z)$$

$$A = 3\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z \qquad (3, 2, 4)$$

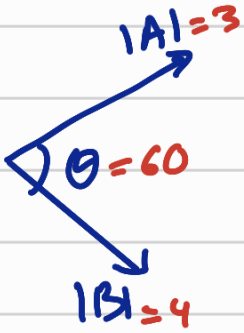
ضرب المتجهات

ضرب قياسي
Scalar Product

$\vec{A}\cdot\vec{B} =$ الناتج عدد

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A} = 3a_x + 2a_y - 5a_z$$
$$\vec{B} = 2a_x - a_y + 2a_z$$
$$\vec{A}\cdot\vec{B} = 3\cdot 2 + 2(-1) + (-5)2$$
$$= 6 - 2 - 10$$
$$= -6$$

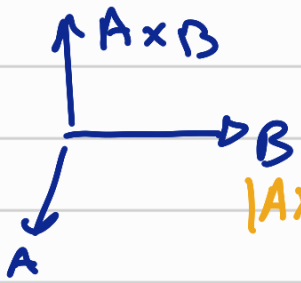


$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta$$
$$A\cdot B = 3\cdot 4\cdot\cos 60$$
$$= 6$$

$$\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1$$
$$\hat{a}_x\cdot a_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0$$

الضرب الاتجاهي
Cross product

$\vec{A}\times\vec{B} =$ الناتج متجه متعامد عمودي على المتجهين

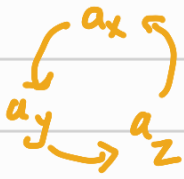


$$|A\times B| = |A||B|\sin\theta$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} = 2\hat{a}_x + a_y - 2\hat{a}_z$$
$$\vec{B} = 3\hat{a}_x + \hat{a}_y + \hat{a}_z$$

$$\begin{vmatrix} a_x & a_y & a_z \\ 2 & 1 & -2 \\ 3 & 1 & 1 \end{vmatrix}$$



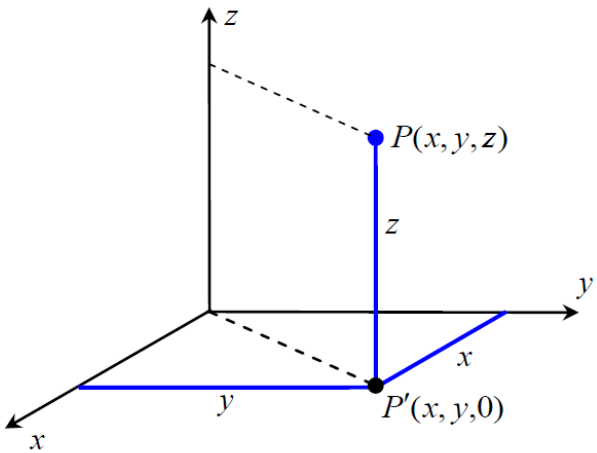
$$= (1+2)\hat{a}_x - (2+6)\hat{a}_y$$
$$(2-3)\hat{a}_z$$
$$= 3\hat{a}_x - 8a_y - \hat{a}_z$$

$$a_x\times a_y = a_y\times a_z = a_z\times a_x = 0$$
$$\hat{a}_x\times a_y = a_z \qquad a_y\times a_z = a_x$$

2- Coordinate Systems and Transformations

2-1- Cartesian Coordinates (*x,y,z*)

A point P can be represented as *(x,y,z)*



The ranges of the coordinate variables x, y, and z are

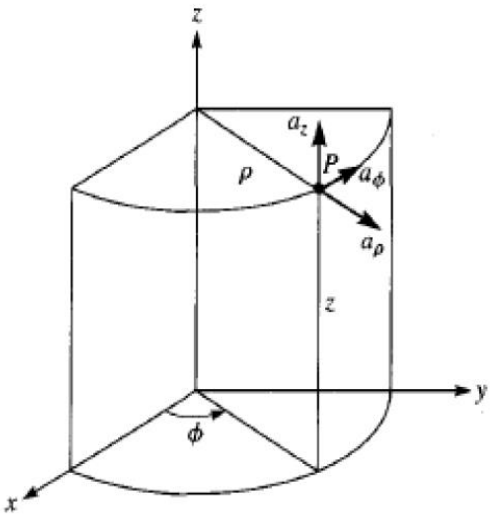
$$-\infty < x < +\infty$$
$$-\infty < y < +\infty$$
$$-\infty < z < +\infty$$

A vector $\vec{A}$ in Cartesian coordinates can be written as

$$\vec{A} = (A_x, A_y, A_z) \quad or \quad \vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

2-2- Circular Cylindrical Coordinates (ρ,ϕ,z)

A point P in cylindrical coordinates is represented as (ρ,ϕ,z) and is as shown in Figure 3. ρ is the radius of the cylinder passing through P or the radial distance from the z-axis. Φ called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system.



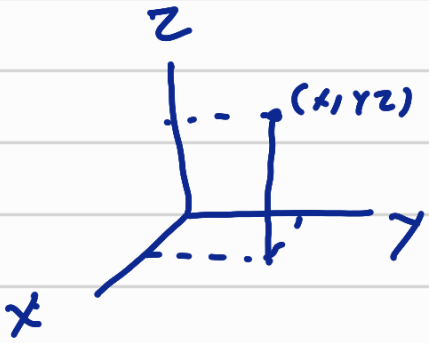
$$0 \le \rho < \infty$$
$$0 \le \phi < 2\pi$$
$$-\infty < z < \infty$$

Cartesian coordinate system

(x, y, z)

$$\bar{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

(A_x, A_y, A_z)

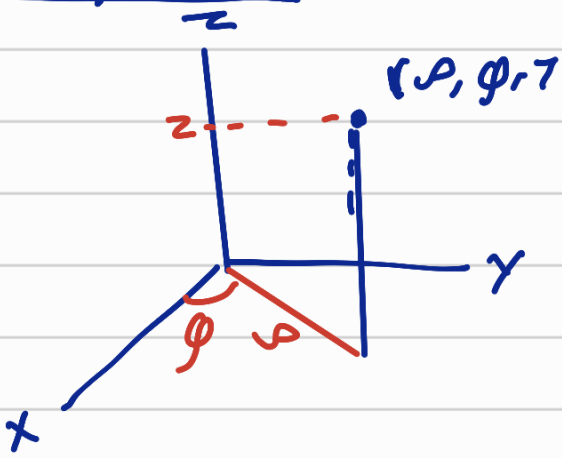


cylindrical coordinate system

(ρ, φ, z)

$$A = A_\rho \hat{a}_\rho + A_\varphi \hat{a}_\varphi + A_z \hat{a}_z$$

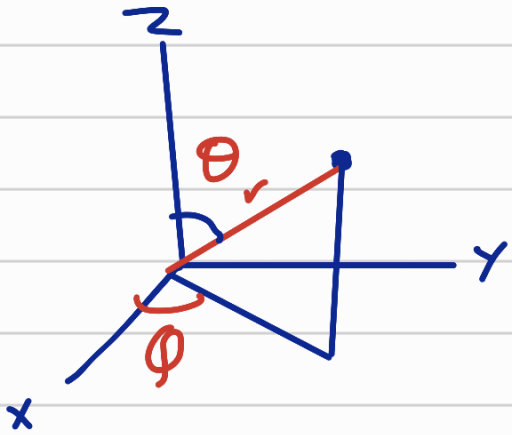
φ : Azimuthal



spherical coordinate system

(r, φ, θ)

θ : Colatitude



differential element in length

احداثيات	المتجه	$dr = dl$
(x, y, z)	$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$	$dr = dl = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$
(r, φ, z)	$\bar{r} = \rho\hat{a}_\rho + z\hat{a}_z$	$dr = dl = d\rho\hat{a}_\rho + \rho d\varphi\hat{a}_\varphi + dz\hat{a}_z$
(r, φ, θ)	$r = r\hat{a}_r$	$dr = dl = dr\hat{a}_r + r d\theta\hat{a} + r\sin\theta d\varphi\hat{a}_\varphi$

2-3- Spherical Coordinates System

A point P can be represented as (r, θ, ϕ) and is illustrated in Figure 6. From Figure 6, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitude) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates).

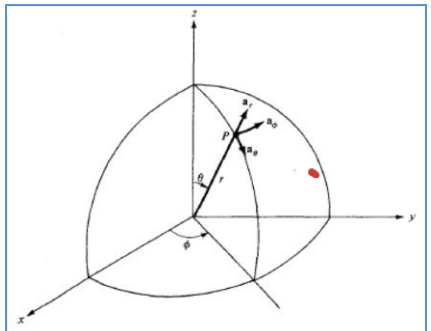
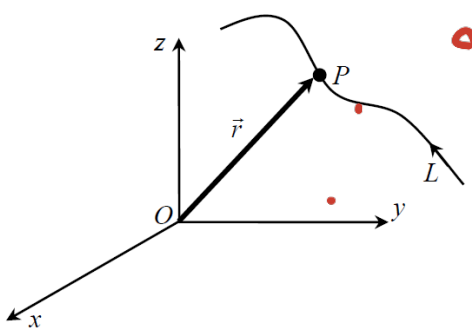


Figure 6.

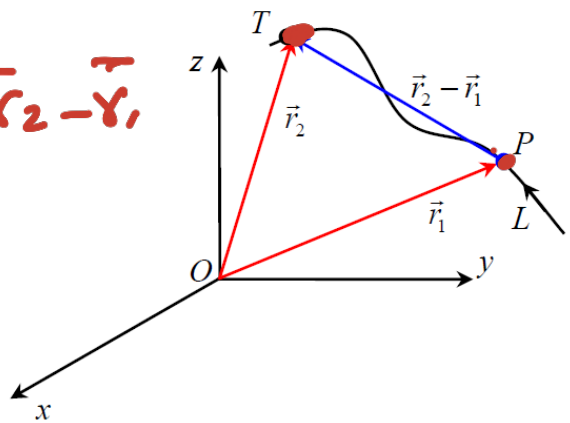
$$0 \le r \langle \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \langle 2\pi$$

3- Vector position and Differential element in length

Consider a point P located in space, and consider the direct arrow extending from the origin to this point P. This arrow is known as the position vector $\vec{r}$. A position vector is an alternative way to denote the location of a point P in space.

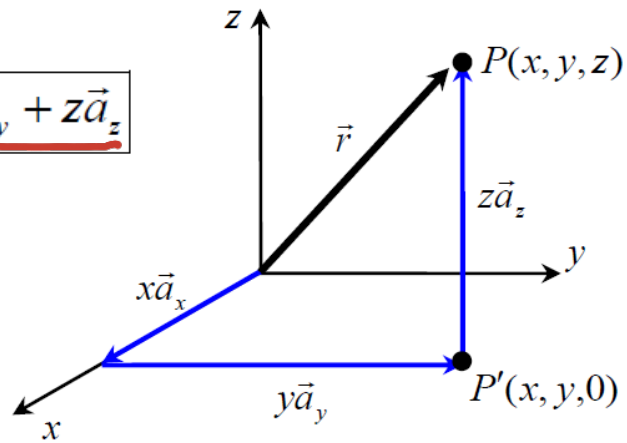


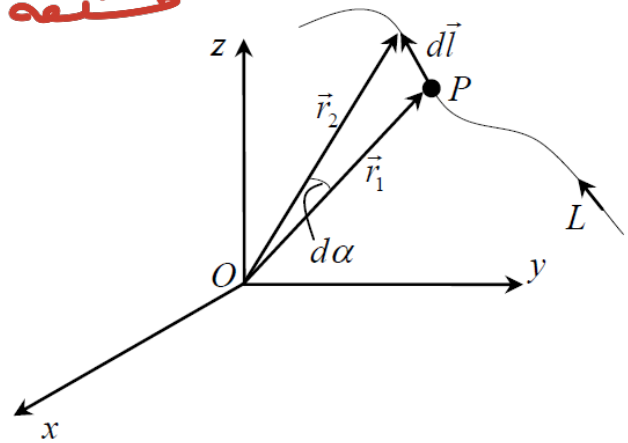
$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$



3. 1- Vector position, $\vec{r}$, and differential element in length, $d\vec{l}$, in Cartesian coordinates system:

$$\boxed{\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}$$





when $\vec{r}_2 - \vec{r}_1 = \Delta\vec{r} \to 0$ (*i.e.*, $\Delta\vec{r} = d\vec{r} = d\vec{l}$),

(1) Differential displacement is given by

$$\boxed{d\vec{r} = d\vec{l} = dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z}$$

(2) Differential normal area is given by

$$\boxed{\begin{aligned} d\mathbf{S} = {} & dy\,dz\,\mathbf{a}_x \\ & dx\,dz\,\mathbf{a}_y \\ & dz\,dy\,\mathbf{a}_z \end{aligned}}$$

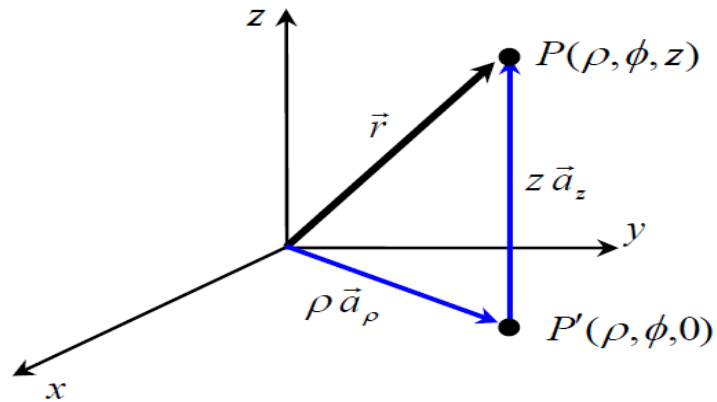
(3) Differential volume is given by

$$\boxed{dv = dx\,dy\,dz}$$

3. 2- Vector position, $\vec{r}$, and differential element in length, $d\vec{l}$, in cylindrical coordinates system:

$$\boxed{\vec{r} = \rho\,\vec{a}_\rho + z\,\vec{a}_z}$$

$$\boxed{d\vec{r} = d\vec{l} = d\rho\,\vec{a}_\rho + \rho\,d\phi\,\vec{a}_\phi + dz\,\vec{a}_z}$$



(2) Differential normal area is given by

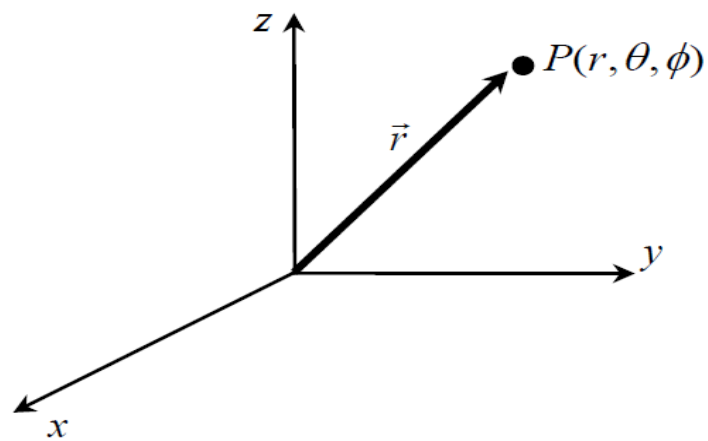
$$\boxed{\begin{aligned} d\mathbf{S} = {} & \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ & d\rho\, dz\, \mathbf{a}_\phi \\ & \rho\, d\phi\, d\rho\, \mathbf{a}_z \end{aligned}}$$

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz}$$

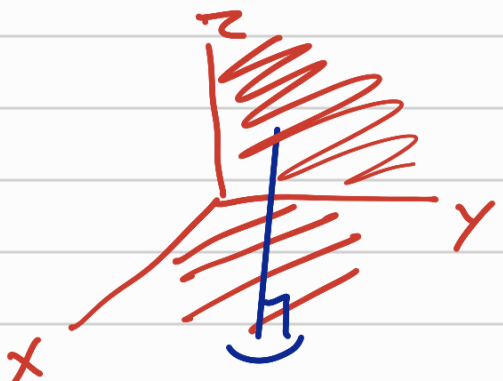
3. 3- Vector position, $\vec{r}$, and differential element in length, $d\vec{l}$, in spherical coordinates system:

$$\boxed{\vec{r} = r\,\vec{a}_r}$$

$$\boxed{d\vec{r} = d\vec{l} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi}$$



احداثيات	المتجه	Differential normal area, volume
(x, y, z)	$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$	$ds = dx\,dy\,\hat{a}_z$ $ds = dy\,dz\,\hat{a}_x$ $ds = dx\,dz\,\hat{a}_y$ $dv = dx\,dy\,dz$
(r, ϕ, z)	$\bar{r} = \rho\hat{a}_\rho + z\hat{a}_z$	$ds = \rho\,d\phi\,dz\,\hat{a}_\rho$ $ds = d\rho\,dz\,\hat{a}_\phi$ $ds = \rho\,d\phi\,d\rho\,\hat{a}_z$ $dv = \rho\,d\rho\,d\phi\,dz$



4. Line integral

تكامل خطي

By a line we mean the path along a curve in space. We shall use terms such as line, curve, path and contour interchangeably.

$\int_L A \cdot dl = \int_a^b |A| \cos\theta \, dl$

The line integral $\boxed{\int_L \vec{A} \cdot d\vec{l}}$ is the integral of the tangential component of $\vec{A}$ along curve L.

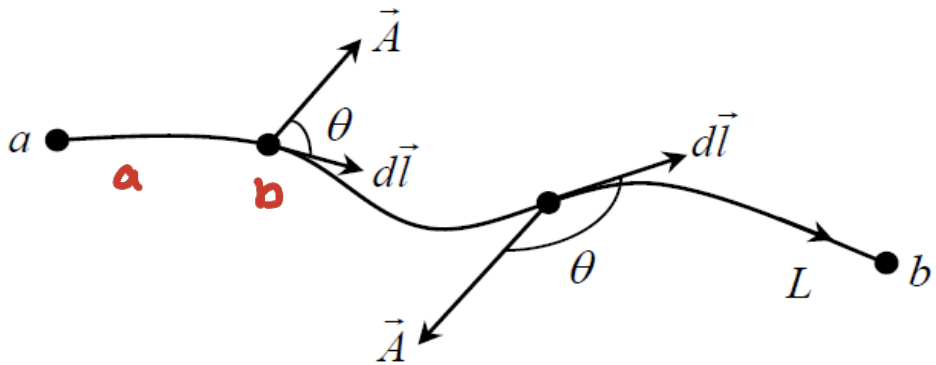


Figure. Path of integration of vector field $\vec{A}$.

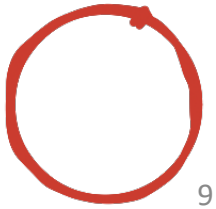
Given a vector field $\vec{A}$ and a curve L, we define the integral $\int_L \vec{A} \cdot d\vec{l} = \int_a^b |\vec{A}| \cos\theta \, dl$ as the integral of $\vec{A}$ around L (see Figure).

If the path of integration is a closed curve $\int_L \vec{A} \cdot d\vec{l}$ becomes a closed line integral $\oint_L \vec{A} \cdot d\vec{l}$ which is called the circulation of $\vec{A}$ around L.

دوران المتجه حول l

$\oint A \cdot dl$

$\oint A\cos\theta \, dl$



5. Surface integral تكامل سطحي (التدفق) Flux ψ

Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or the flux of $\vec{A}$ through S (see figure) as

$$\boxed{\psi = \int_S |\vec{A}| \cos\theta \, dS = \int_S \vec{A} \cdot \vec{a}_n dS}$$

$$\psi = \int \bar{A} \cdot \bar{a}_n \, ds = \int |A| \cos\theta \, ds$$

or simply

$$\boxed{\psi = \int_S \vec{A} \cdot d\vec{S}}$$

where at any point on S, $\vec{a}_n$ is the unit normal to S.

$\hat{a}_n$: العمودي

سطح مغلق

For a closed surface, the surface integral becomes

$$\boxed{\psi = \oint_S \vec{A} \cdot d\vec{S}}$$

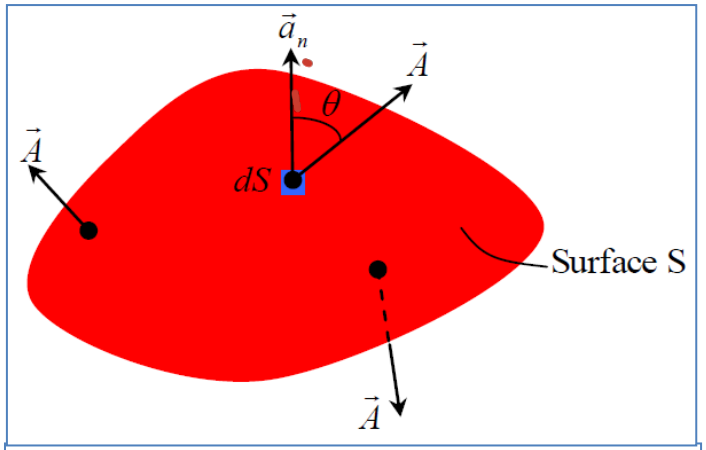


Figure. The flux of a vector field $\vec{A}$ through surface S

which is referred to as the net outward flux of $\vec{A}$ from S.

التكامل عبر سطح مغلق :- التدفق الكلي الخارج من السطح

as the *volume integral* of the scalar ρ_v over the volume v.

$$\int_v \rho_v \, dv$$

تكامل الحجم

كثافة حجمية

PRACTICE EXERCISE 3.2

$\int A \cdot dl$

Calculate the circulation of

(ρ, ϕ, z)

$$\mathbf{A} = \rho \cos \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in Figure 3.11.

Answer: 1.

$\hat{a}_\rho \cdot \hat{a}_\rho = 1$

$a_\rho \cdot a_\phi = 0$

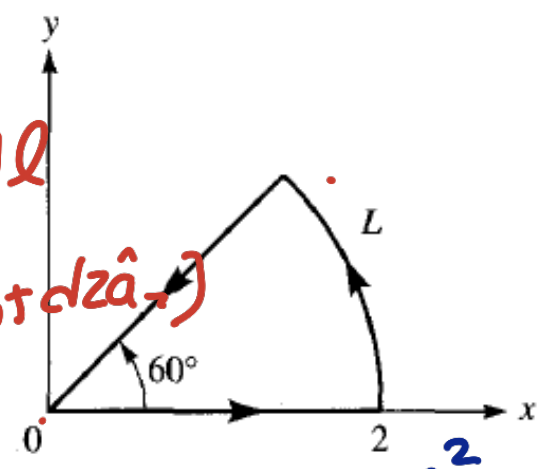
$\hat{a}_\rho \cdot \hat{a}_z = 0$

$\int A \cdot dl$

0

$\int (\rho \cos\phi \, a_\rho + z \sin\phi \, a_z) \cdot dl$

$\int_L (\rho \cos\phi \, \hat{a}_\rho) \cdot (d\rho \, \hat{a}_\rho + \rho \, d\phi \, \hat{a}_\phi + dz \, \hat{a}_z)$



$\int_0^2 \rho \cos\phi \, d\rho + 0 + 0 = \cos\phi \int_0^2 \rho \, d\rho = \cos 60 \left. \frac{\rho^2}{2} \right]_0^2$

$\frac{1}{2} \frac{1}{2} [2^2 - 0^2] = \frac{1}{4} [4] = 1$

4. DEL OPERATOR

∇ del

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

This vector differential operator, otherwise known as the *gradient operator,* is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Gradiunt of scalar ∇V

تدل على اتجاه اكبر معدل للزيادة لهذه الكميه الفيزيائيه

* احداثيات كارتيزيه (x, y, z)

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$$

* احداثيات اسطوانيه (ρ, φ, z)

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi + \frac{\partial V}{\partial z}\hat{a}_z}$$

* احداثيات كرويه

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi}$$

- The del Operator in cylindrical system from the definition of the gradient:

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\vec{a}_\phi + \frac{\partial}{\partial z}\vec{a}_z}$$ in cylindrical system

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\vec{a}_\phi + \frac{\partial f}{\partial z}\vec{a}_z}$$ in cylindrical system

- The del Operator in spherical system from the definition of the gradient:

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\vec{a}_\phi}$$ in spherical system

and

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{a}_\phi}$$ in spherical system

EXAMPLE 3.3

Find the gradient of the following scalar fields:

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

Solution:

(a) $$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= 2e^{-z}\cos 2x \cosh y\, \mathbf{a}_x + e^{-z}\sin 2x \sinh y\, \mathbf{a}_y - e^{-z}\sin 2x \cosh y\, \mathbf{a}_z$$

(b) $$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$
$$= 2\rho z \cos 2\phi\, \mathbf{a}_\rho - 2\rho z \sin 2\phi\, \mathbf{a}_\phi + \rho^2 \cos 2\phi\, \mathbf{a}_z$$

(c) $$\nabla W = \frac{\partial W}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r \sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$
$$= 10 \sin^2\theta \cos\phi\, \mathbf{a}_r + 10 \sin 2\theta \cos\phi\, \mathbf{a}_\theta - 10 \sin\theta \sin\phi\, \mathbf{a}_\phi$$

Find the gradiant

a) $V = e^{-z} \sin 2x \cos hy$

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$\nabla V = 2e^{-z}\cos 2x \cosh y\, \hat{a}_x + e^{-z}\sin 2x \sinh y\, \hat{a}_y - e^{-z}\sin 2x \cosh y\, \hat{a}_z$$

(b) $U = \rho^2 \cos 2\varphi$

$$\nabla U = \frac{\partial U}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \varphi}\hat{a}_\varphi + \frac{\partial U}{\partial z}\hat{a}_z$$

$$\nabla U = 2\rho z \cos 2\varphi\, \hat{a}_\rho - 2z\sin 2\varphi\, \hat{a}_\varphi + \rho^2 \cos 2\varphi\, \hat{a}_z$$

$$\nabla U = 2\rho z \cos 2\varphi\, \hat{a}_\rho - 2z\sin 2\varphi\, \hat{a}_\varphi + \rho^2 \cos 2\varphi\, \hat{a}_z$$

$\boxed{2\sin x \cos x = \sin 2x}$

(c) $W = 10 r \sin^2\theta \cos\varphi$

$$\nabla W = \frac{\partial W}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \varphi}\hat{a}_\varphi$$

$$\nabla W = 10\sin^2\theta\cos\varphi\, \hat{a}_r + \frac{2}{\cancel{r}}10\cancel{r}\sin\theta\cos\varphi\, \hat{a}_\theta - \frac{1}{r\sin\theta}10r\sin^2\theta\sin\varphi\, \hat{a}_\varphi$$

$$\nabla W = 10\sin^2\theta\cos\varphi\, \hat{a}_r + 10\sin 2\theta\cos\varphi\, \hat{a}_\theta - 10\sin\theta\sin\varphi\, \hat{a}_\varphi$$

Fields

Many physical quantities have different values at different points in space. For example, the temperature in the kitchen is different at different points: high near an electric furnace, low near an open window, and so on. The electric field around a point charge is large near the charge and decreases as we go away from the charge. In all these examples there is a particular region of space which is of interest for the problem at hand; at every point of this region some physical quantity has a value. The term field is used to mean both the region and the value of the physical quantity in the region. If the physical quantity is a scalar (for example, the temperature), we speak of a scalar field. If the quantity is a vector (for example, electric field), we speak of a vector field.

Directional derivative; Gradient

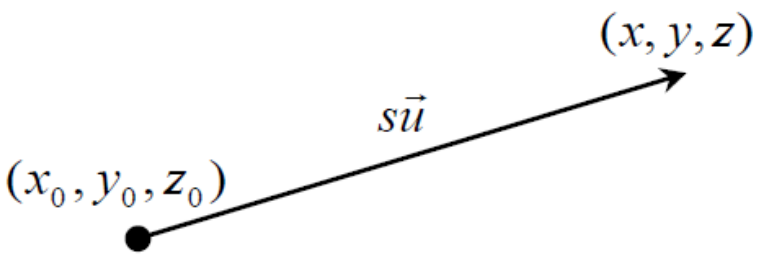
Suppose that we know the temperature $T(x,y,z)$ at every point of a room. Starting at a given point (x_0,y_0,z_0) we could ask for the rate of change of the temperature with distance (in degrees per centimeter) as we move away from the starting point (x_0,y_0,z_0). The rate of change of temperature with distance depends upon the direction in which we move. In symbols, we want to find the limiting value of $\frac{\Delta T}{\Delta s}$ where Δs is an element of distance in a given direction, and ΔT is the corresponding change in temperature; we write the directional derivative as $\frac{dT}{ds}$.

Let us now state and solve the general problem of finding a directional derivative.

$$(x_0, y_0, z_0) \xrightarrow[s\hat{u}]{s} (x, y, z)$$

We are given a scalar field, that is, a function $f(x,y,z)$. We want to find $\frac{df}{ds}$, the rate of change of f with distance, at a given point (x_0,y_0,z_0) and in a given direction. Let $\vec{u} = u_1\vec{a}_x + u_2\vec{a}_y + u_3\vec{a}_z$ be a unit vector in the given direction. In figure, we start at (x_0,y_0,z_0) and go a distance s in the direction $\vec{u}$ to the point (x,y,z); the vector joining these points is $s\vec{u}$. Then, $(x,y,z)-(x_0,y_0,z_0) = s\vec{u} = s(u_1,u_2,u_3)$ or $\begin{cases} x = x_0 + u_1 s \\ y = y_0 + u_2 s \\ z = z_0 + u_3 s \end{cases}$

متجه الوحدة



متجه الوحدة (طوله 1)
ويعبر فقط عن الاتجاه

$$\frac{df}{ds} = \vec{\nabla} f \cdot \vec{u}$$

$\frac{df}{ds}$ = directional derivative

where

$$\boxed{\vec{\nabla} f \equiv \frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z}$$

in Cartesian system

$\boxed{\nabla f \cdot u} = |\nabla f| \cos\theta$

$\frac{\text{المتجه}}{\text{مقدارة}}$ = متجه الوحدة = u

and

$$\vec{u} = \frac{\vec{r}(x,y,z) - \vec{r}(x_0,y_0,z_0)}{\left|\vec{r}(x,y,z) - \vec{r}(x_0,y_0,z_0)\right|} = \boxed{\frac{\Delta\vec{r}}{\left|\Delta\vec{r}\right|}}$$

$r = 3a_x + 4a_y$

$\hat{u} = \frac{3a_x + 4a_y}{5}$

When (x,y,z) approaches (x_0,y_0,z_0) then $\vec{u} = \frac{d\vec{r}}{|d\vec{r}|} = \boxed{\frac{d\vec{r}}{ds}}$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \vec{u} = \vec{\nabla} f . \frac{d\vec{r}}{ds}$$

$\frac{df}{ds} = \vec{\nabla} f \cdot \vec{u}$ is the directional derivative (ds is an element of distance in the direction $\vec{u}$)

$$\frac{df}{ds} = \left|\vec{\nabla} f\right| \underbrace{\left|\vec{u}\right|}_{1} \cos\theta = \left|\vec{\nabla} f\right| \cos\theta$$, where θ is the angle between $\vec{u}$ and $\vec{\nabla} f$.

Thus $\frac{df}{ds}$ is the projection of $\vec{\nabla} f$ on direction $\vec{u}$ (see the left figure below). We find the largest value of $\frac{df}{ds}$ (namely $\left|\vec{\nabla} f\right|$) if we go in the direction of $\vec{\nabla} f$ ($\theta = 0$.)

$$\boxed{\left(\frac{df}{ds}\right)_{\max} = \left|\vec{\nabla} f\right|}$$

$\theta = 0$

$\cos\theta = 1$

$\left.\frac{df}{ds}\right|_{\max} = |\nabla f|$

The gradient of a scalar field $f(x,y,z)$ is a vector field $\vec{\nabla} f$ that points in the direction of the **greatest** rate of increase of the scalar field, and whose magnitude is that rate of increase.

EXAMPLE 3.4

Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:

$$\nabla W = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$

$$= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$

Hence,

$$\frac{dW}{dl} = \nabla W \cdot \mathbf{a}_l = (4, -8, -2) \cdot \frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

Directional derevative $= \frac{df}{ds}$

rate of change of f with distance S

$$\frac{df}{ds} = \nabla f \cdot \bar{u} = |\nabla f| \cos\theta$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}a_z$$

$$\hat{u} = \frac{\Delta r}{|\Delta r|}$$

اكبر معدل للتغير عندما تكون $\theta = 0$

$$\frac{df}{ds} = |\nabla f| \Rightarrow$$

وهو نفس تعريف ∇f

EXAMPLE 3.4 Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

a)

$$\nabla W = \frac{\partial W}{\partial x}\hat{a}_x + \frac{\partial W}{\partial y}a_y + \frac{\partial W}{\partial z}a_z$$

$$\nabla W = (2xy^2 + yz)\hat{a}_x + (2yx^2 + xz)\hat{a}_y + (xy)\hat{a}_z$$

$$\nabla W = (2(2(-1) + 0)a_x + (2(-1)(4) + 0)a_y + ((2)(-1)a_z$$

$$\nabla W = -4a_x + -8a_y - 2a_z$$

b)

$$\frac{dW}{dl} = \nabla W \cdot \hat{u}_l \qquad \bar{u}_l = \frac{\vec{l}}{|l|} = \frac{3a_x + 4a_y + 12a_z}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$u = \frac{1}{13}(3a_x + 4a_y + 12a_z)$$

$$\frac{dW}{dl} = \nabla W \cdot \bar{u} = \frac{1}{13}(-4a_x - 8a_y - 2a_z) \cdot (3a_x + 4a_y + 12a_z) = \frac{-44}{3}$$

5- Divergence of a vector and divergence theorem

The divergence of a vector field $\vec{A}$ at a given point P is the net outward flux over a closed surface S per unit volume as the volume shrinks about P.

$$\boxed{div\vec{A} = \vec{\nabla}\cdot\vec{A} \equiv \lim_{\Delta v\to 0}\frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta v}}$$

$$div A = \nabla\cdot\vec{A} = \lim_{\Delta v\to 0}\frac{\oint A\cdot ds}{\Delta v}$$

where Δv is the volume enclosed by the closed surface S in which P is located.

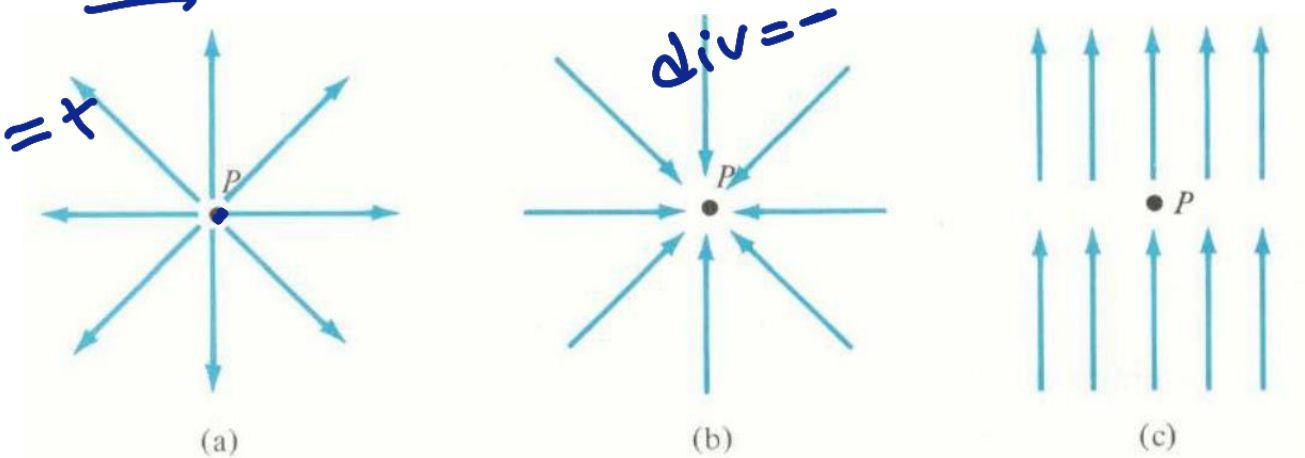
-The physical significance of the divergence of a vector field

The divergence of the vector field $\vec{A}$ at a point P is a measure of how much the field diverges or spreads out from that point.

div = +

div = −

div = zero



div f
⇒ تباعد

Figure. Illustration of the divergence of a vector field at P; (a) positive divergence, (b) negative divergence, (c) zero divergence.

The divergence of $\vec{A}$ at point $P(x_0,y_0,z_0)$ in Cartesian system is given by

$\nabla \cdot \vec{A}$ الكارتيزية

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$f = 2x + y + z$

$\nabla \cdot f = 2 + 1 + 1 = 4$

$f = 2x a_x + y a_y + 3z a_z$

$\nabla \cdot f = 2 + 1 + 3 = 6$

The divergence of $\vec{A}$ at point $P(x_0,y_0,z_0)$ in cylindrical coordinates

$\nabla \cdot \vec{A}$ الاسطوانية

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

The divergence of $\vec{A}$ at point $P(x_0,y_0,z_0)$ in spherical coordinates

$\nabla \cdot \vec{A}$ الكروية

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

- Properties of the divergence of a vector field:

1- $\vec{\nabla} \cdot \vec{A}$ is a scalar field. قياسية

div لكمية قياسية ليس لها معنى

2- The divergence of a scalar field V, $\vec{\nabla} \cdot V$, makes no sense

3- $\vec{\nabla} \cdot (\vec{A} + \vec{B}) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$ يجوز التوزيع

4- $\vec{\nabla} \cdot (V\vec{A}) = V\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla} V$

EXAMPLE 3.6

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

a) $\nabla\cdot P = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$

$= 2xyz + 0 + x$

$\boxed{\nabla\cdot P = 2xyz + x}$

Solution:

(a) $$\nabla\cdot\mathbf{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$

$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$= 2xyz + x$$

$\nabla\cdot Q = \frac{1}{\rho}\frac{\partial \rho Q_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_z}{\partial z}$

(b) $$\nabla\cdot\mathbf{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}Q_\phi + \frac{\partial}{\partial z}Q_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= 2\sin\phi + \cos\phi$$

$\nabla\cdot Q = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho^2\sin\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$

$= 2\sin\phi + 0 + \cos\phi$

- Divergence theorem:

$$\oint \vec{A}\cdot ds = \int_v \nabla\cdot\vec{A}\, dv$$

From the definition of the divergence of $\vec{A}$, it is not difficult to expect that

$$\boxed{\oint_s \vec{A}\cdot d\vec{S} = \int_v (\vec{\nabla}\cdot\vec{A})\, dv}$$ divergence theorem

The divergence theorem states that the total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$

$$\oint_s G\cdot ds$$

$$G = 10e^{-2z}\rho\, a_\rho + 10e^{-2z} a_z$$

EXAMPLE 3.7 If $\mathbf{G}(r) = 10e^{-2z}(\rho\mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1$, $0 \le z \le 1$. Confirm the result using the divergence theorem.

$$\oint G\cdot ds = \int_v \nabla\cdot G\, dv$$

Solution:

$$\Psi = \oint_S G\cdot d\mathbf{S} = \int_v (\nabla\cdot\mathbf{G})\, dv$$

$$\nabla\cdot G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho G_\rho + \frac{1}{\rho}\frac{\partial G_\phi}{\partial\phi} + \frac{\partial G_z}{\partial z}$$

$$\nabla\cdot\mathbf{G} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}G_\phi + \frac{\partial}{\partial z}G_z$$

$$\nabla\cdot G = \frac{1}{\rho}\frac{\partial}{\partial\rho}10e^{-2z}\rho^2 + \frac{1}{\rho}\frac{\partial}{\partial\phi}0 + \frac{\partial}{\partial z}10e^{-2z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2 10e^{-2z}) - 20e^{-2z} = 0$$

$$\nabla\cdot G = 20e^{-2z} + 0 - 20e^{-2z} = 0$$

$$\Psi = \int_v (\nabla\cdot G)\, dv = 0$$

$$\int \nabla\cdot G\, dv = \int 0\, dv = 0$$

6- Curl of a vector and Stokes's theorem

$\nabla \times \bar{A}$

The curl of $\vec{A}$ (curl $\vec{A}$ or $\vec{\nabla} \times \vec{A}$) is a vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$\boxed{\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} \equiv \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{l}}{\Delta S} \right)_{\max} \vec{a}_n}$$

curle

$$\nabla \times \bar{A} = \left(\lim_{\Delta S \to 0} \frac{\oint A \cdot dl}{\Delta S} \right)_{max} \hat{a}_n$$

ΔS: area bounded by the closed path L

$\vec{a}_n$: unit vector normal to the surface ΔS and is determined using the right-hand rule.

-The physical significance of the curl of a vector field

الدوران

1- The curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs.

2- The curl of a vector field $\vec{A}$ at a point P may be regarded as a measure of the circulation or how much the field curls around P.

In Cartesian coordinates $\vec{\nabla} \times \vec{A}$ is found using

$$\boxed{\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}}$$

or

$$\boxed{\vec{\nabla} \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\vec{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{a}_z}$$

We obtain $\vec{\nabla} \times \vec{A}$ in cylindrical coordinates as

$$\vec{\nabla}\times\vec{A}=\frac{1}{\rho}\begin{vmatrix}\vec{a}_\rho & \rho\vec{a}_\phi & \vec{a}_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

$$\frac{1}{\rho}\begin{vmatrix}a_\rho & \rho a_\phi & a_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

or

$$\vec{\nabla}\times\vec{A}=\left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right]\vec{a}_\rho-\left[\frac{\partial A_z}{\partial\rho}-\frac{\partial A_\rho}{\partial z}\right]\vec{a}_\phi+\frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi}\right]\vec{a}_z$$

and in spherical coordinates as

$$\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}r\sin\theta A_\phi-\frac{\partial rA_\theta}{\partial\phi}\right]a_r$$

or

$$\vec{\nabla}\times\vec{A}=\frac{1}{r\sin\theta}\left[\frac{1}{\rho}\frac{\partial(A_\phi\sin\theta)}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi}\right]\vec{a}_r$$
$$-\frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r}-\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}\right]\vec{a}_\theta+\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right]\vec{a}_\phi$$

- Properties of the curl of a vector field: خصائص الدوران

1- The curl of a vector field is another vector field. النتيجه هي متجه

2- The curl of a scalar field V, $\vec{\nabla}\times V$, makes no sense. الدوران بي مجهول قياسه ليس له معنى

3- $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

a b c bac − cab + cab − bac

4- $\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}$

5- $\vec{\nabla}\times(V\vec{A})=V\vec{\nabla}\times\vec{A}+\vec{\nabla}V\times\vec{A}$

6- The divergence of the curl of a vector field vanishes, that is, $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0}$.

7- The curl of the gradient of a scalar vanishes, that is, $\boxed{\vec{\nabla}\times\vec{\nabla}V=0}$

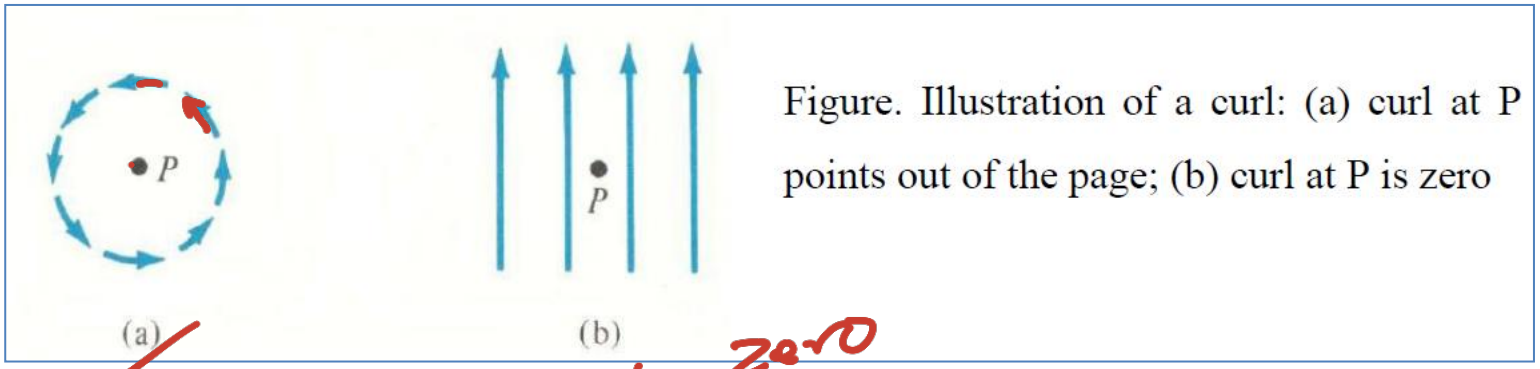


Figure. Illustration of a curl: (a) curl at P points out of the page; (b) curl at P is zero

خارج الورقة

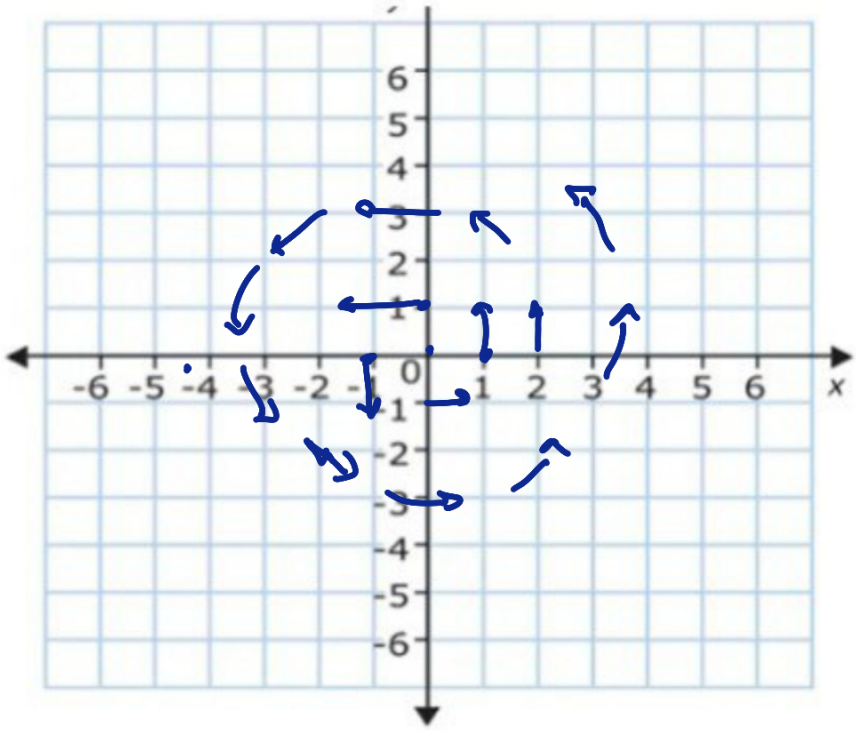
curl = zero

$\vec{v}_a = -y\hat{x} + x\hat{y}$

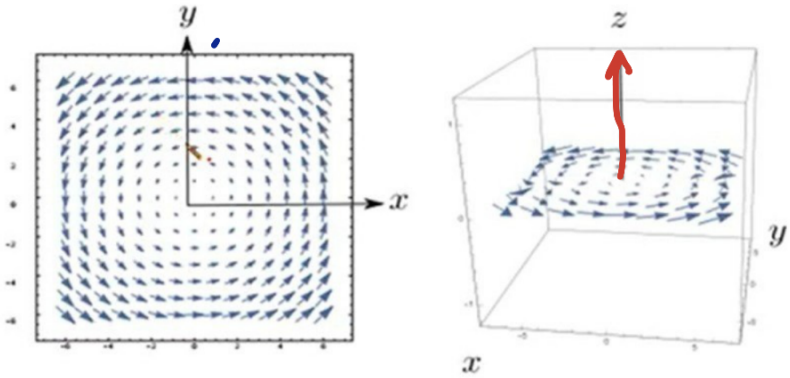
$(1,0) \quad V_a = \hat{y}$

$(0,1) \quad V_a = -\hat{x}$

$(-1,0) \quad V_a = -y$



$\vec{v}_a = -y\hat{x} + x\hat{y}$



$$\nabla \times V = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$a_x \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & 0 \end{vmatrix} - a_y \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -y & 0 \end{vmatrix} + a_z \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix}$$

$$a_x\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}x\right) - a_y\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}-y\right) + a_z\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}-y\right)$$

$$(0 - 0)a_x - (0 - 0)a_y + (1 - -1)a_z$$

$$= 2a_z$$

EXAMPLE 3.8

Determine the curl of the vector fields in Example 3.6.

Solution:

(a)
$$\nabla \times \mathbf{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z$$
$$= (0 - 0)\,\mathbf{a}_x + (x^2y - z)\,\mathbf{a}_y + (0 - x^2z)\,\mathbf{a}_z$$
$$= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z$$

(b)
$$\nabla \times \mathbf{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right]\mathbf{a}_z$$
$$= \left(\frac{-z}{\rho}\sin\phi - \rho^2\right)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(3\rho^2 z - \rho\cos\phi)\mathbf{a}_z$$
$$= -\frac{1}{\rho}(z\sin\phi + \rho^3)\mathbf{a}_\rho + (3\rho z - \cos\phi)\mathbf{a}_z$$

EXAMPLE 3.6

Determine the ~~divergence~~ curl of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \underbrace{\rho \sin\phi}_{A_\rho}\,\mathbf{a}_\rho + \underbrace{\rho^2 z}_{A_\phi}\,\mathbf{a}_\phi + \underbrace{z\cos\phi}_{A_z}\,\mathbf{a}_z$

$$\nabla \times P = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 0 & xz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} 0\right]\hat{a}_x - \left[\frac{\partial}{\partial x} xz - \frac{\partial}{\partial z} x^2yz\right]\hat{a}_y + \left[\frac{\partial}{\partial x} 0 - \frac{\partial}{\partial y} x^2yz\right]a_z$$

$$0\,\hat{a}_x - [z - x^2y]\,a_y + [x^2z]\,a_z$$

$$\nabla \times P = [x^2y - z]\,a_y + x^2z\,a_z$$

$$\nabla \times Q = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho\sin\phi & \rho^2 z & z\cos\phi \end{vmatrix}$$

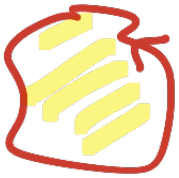
$$\frac{1}{\rho}\left[\frac{\partial}{\partial \phi} z\cos\phi - \frac{\partial}{\partial z}\rho^2 z\right]a_\rho - \frac{1}{\rho}\left[\frac{\partial}{\partial \rho} z\cos\phi - \frac{\partial}{\partial z}\rho\sin\phi\right]a_\phi$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\rho^2 z - \frac{\partial}{\partial \phi}\rho\sin\phi\right]a_z$$

$$\frac{1}{\rho}\left[-z\sin\phi - \rho^2\right]a_\rho - \cancel{\frac{1}{\rho}[0-0]\,a_\phi} + \frac{1}{\rho}\left[3\rho^2 z - \rho\cos\phi\right]a_z$$

$$-\frac{1}{\rho}\left[z\sin\phi - \rho^2\right]a_\rho + \left[3\rho z - \cos\phi\right]a_z$$

-Stokes's theorem



From the definition of the curl of $\vec{A}$, we may expect that

$$\boxed{\oint_L \vec{A}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{A})\cdot\vec{a}_n dS}$$

Stokes's theorem

Stokes's theorem states that the circulation of a vector field $\vec{A}$ around a closed path L is equal to the surface integral of $\vec{\nabla}\times\vec{A}$ over the open surface S bounded by L provided that $\vec{A}$ and $\vec{\nabla}\times\vec{A}$ are continuous on S.

$$\oint A\cdot dl = \int (\nabla\times A)\cdot a_n \, ds$$

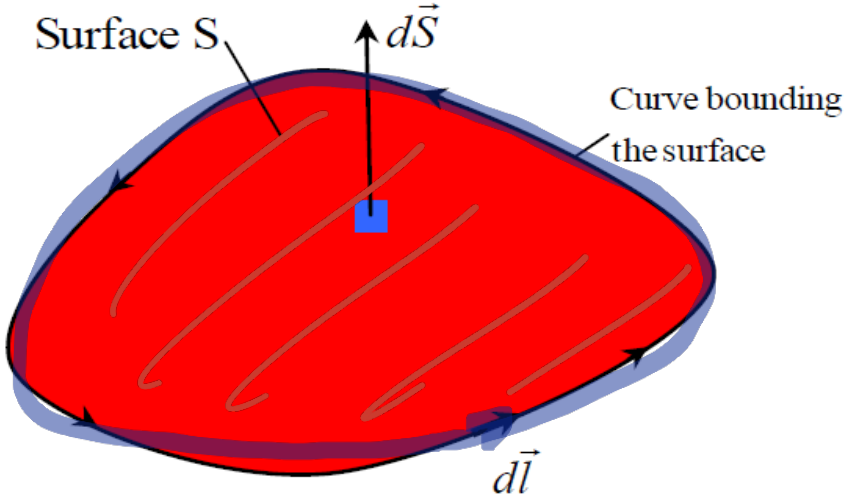


Figure. Determining the sense of $d\vec{l}$ and $d\vec{S}$ involved in Stokes's Theorem

PRACTICE EXERCISE 3.9

Use Stokes's theorem to confirm your result in Practice Exercise 3.2.

Answer: 1.

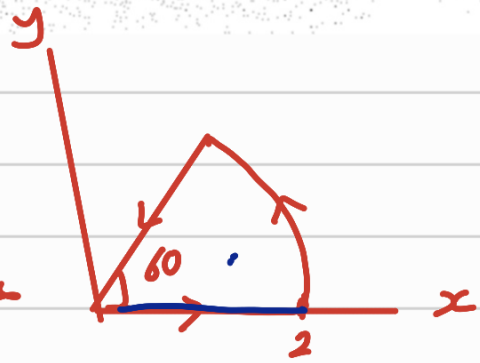
PRACTICE EXERCISE 3.2

Calculate the circulation of

$$\mathbf{A} = \rho \cos \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in Figure 3.11.

Answer: 1.



$$\oint A \cdot dl = \int_s \nabla \times A \cdot ds$$

يمكن حل السؤال باستخدام نظرية ستوكس

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \cos \phi & \rho(0) & z \sin \phi \end{vmatrix}$$

$$\nabla \times A = \frac{1}{\rho}\left[z\cos\phi - 0\right]\hat{a}_\rho - \frac{1}{\rho}\left[0 - 0\right]\rho \cdot \hat{a}_\phi$$

$$\frac{1}{\rho}\left[0 - -\rho \sin\phi\right] a_z$$

$$\nabla \times A = \frac{1}{\rho}\left[z\cos\phi \, a_\rho\right] + \left[\sin\phi \, a_z\right]$$

$$\oint (\nabla \times A) \cdot ds = \int_s \left[\frac{z}{\rho}\cos\phi \, \hat{a}_\rho + \sin\phi \, a_z\right] \cdot \left[\rho \, d\rho \, d\phi \, \hat{a}_z\right]$$

$a_z \cdot a_z = 1$

zero

$$\int (\nabla \times A) \cdot ds = \int_S \rho \sin\varphi \, d\rho \, d\varphi$$

$$= \int_0^2 \rho \, d\rho \int_0^{60} \sin\varphi \, d\varphi$$

$$= \left. \frac{\rho^2}{2} \right]_0^2 \left. -\cos\varphi \right]_0^{60}$$

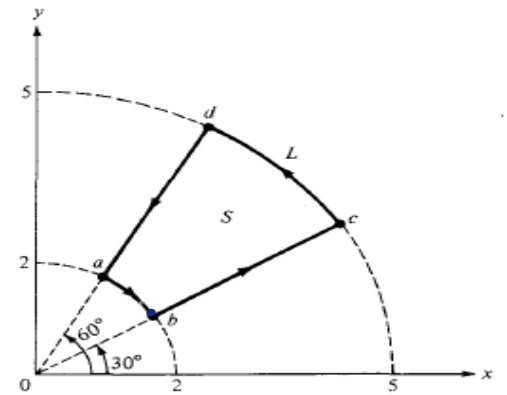
$$\left(\frac{4-0}{2} \right) \left(-\cos 60 - \cos 0 \right)$$

$$2 \left(-\frac{1}{2} + 1 \right)$$

$$2 \left(\frac{1}{2} \right) = 1$$

EXAMPLE 3.9

If $\mathbf{A} = \rho \cos\phi\, \mathbf{a}_\rho + \sin\phi\, \mathbf{a}_\phi$, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.



Solution:

Using Stokes's theorem (because L is a closed path)

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

But $d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$ and

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \mathbf{a}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \mathbf{a}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right]$$

$$= (0 - 0)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(1 + \rho) \sin\phi\, \mathbf{a}_z$$

Hence:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\phi=30^\circ}^{60^\circ} \int_{\rho=2}^{5} \frac{1}{\rho}(1 + \rho) \sin\phi\, \rho\, d\rho\, d\phi$$

$$= \int_{30^\circ}^{60^\circ} \sin\phi\, d\phi \int_2^5 (1 + \rho) d\rho$$

$$= -\cos\phi \Big|_{30^\circ}^{60^\circ} \left(\rho + \frac{\rho^2}{2} \right) \Big|_2^5$$

$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

EXAMPLE 3.9 If $\mathbf{A} = \rho \cos\phi\, \mathbf{a}_\rho + \sin\phi\, \mathbf{a}_\phi$, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.

Solution:

Let

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] \mathbf{A} \cdot d\mathbf{l}$$

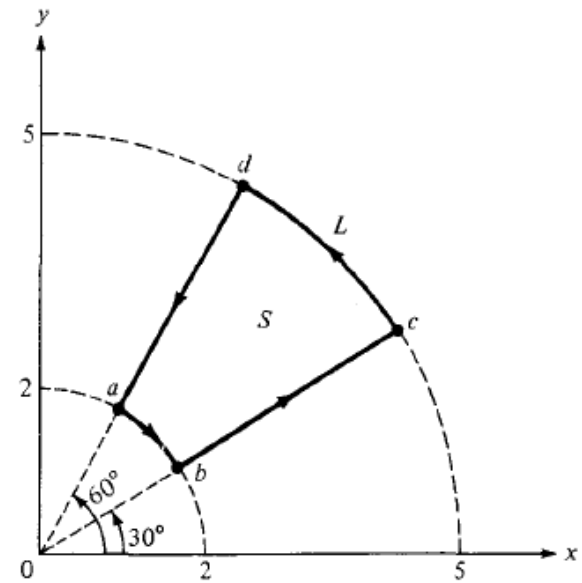


Figure 3.22 For Example 3.9.

where path L has been divided into segments ab, bc, cd, and da as in Figure 3.22.

Along ab, $\rho = 2$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_a^b \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=60^\circ}^{30^\circ} \rho \sin\phi\, d\phi = 2(-\cos\phi) \Big|_{60^\circ}^{30^\circ} = -(\sqrt{3} - 1)$$

Along bc, $\phi = 30^\circ$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho \cos\phi\, d\rho = \cos 30^\circ \frac{\rho^2}{2} \Big|_2^5 = \frac{21\sqrt{3}}{4}$$

Along cd, $\rho = 5$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_c^d \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=30^\circ}^{60^\circ} \rho \sin\phi\, d\phi = 5(-\cos\phi)\Big|_{30^\circ}^{60^\circ} = \frac{5}{2}(\sqrt{3} - 1)$$

Along da, $\phi = 60^\circ$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

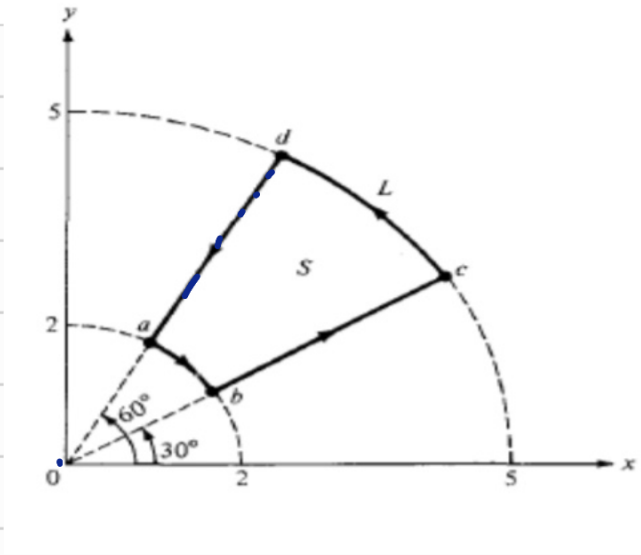
$$\int_d^a \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho \cos\phi\, d\rho = \cos 60^\circ \frac{\rho^2}{2}\Big|_5^2 = -\frac{21}{4}$$

Putting all these together results in

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = -\sqrt{3} + 1 + \frac{21\sqrt{3}}{4} + \frac{5\sqrt{3}}{2} - \frac{5}{2} - \frac{21}{4}$$

$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

EXAMPLE 3.9 If $\mathbf{A} = \rho\cos\phi\,\mathbf{a}_\rho + \sin\phi\,\mathbf{a}_\phi$, evaluate $\oint \mathbf{A}\cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.



$$\oint_L A\cdot dl = \int (\nabla\times A)\cdot ds$$

$$ds = \rho\, d\phi\, d\rho\, a_z$$

$$\nabla\times A = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho\cos\phi & \rho\sin\phi & 0 \end{vmatrix}$$

$$\frac{1}{\rho}[0-0]\,\hat{a}_\rho - \frac{1}{\rho}[0-0]\,a_\phi + \frac{1}{\rho}[\sin\phi + \rho\sin\phi]\,a_z$$

$$\boxed{\nabla\times A = \frac{1}{\rho}(1+\rho)\sin\phi\, a_z}$$

$$\int (\nabla\times A)\cdot ds =$$

$$\hat{a}_z\cdot\hat{a}_z = 1$$

$$\int\left(\frac{1}{\cancel{\rho}}(1+\rho)\sin\phi\,\hat{a}_z\right)\cdot\left(\cancel{\rho}\, d\rho\, d\phi\,\bar{a}_z\right)$$

$$\int \sin\phi\,(1+\rho)\,d\rho\, d\phi$$

$$= \int_{30}^{60} \sin\phi \, d\phi \int_{2}^{5} (1+\rho)\, d\rho$$

$$= \left(-\cos\phi \Big]_{30}^{60} \right) \left(\rho + \frac{\rho^2}{2} \Big]_{2}^{5} \right)$$

$$\left[-\cos 60 + \cos 30 \right] \left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right) \right]$$

$$= 4.941$$

طريقة 2

$$\int A \cdot dl = \int_a^b A \cdot dl + \int_b^c A \cdot dl + \int_c^d A \cdot dl + \int_d^a A \cdot dl$$

$a_\rho \cdot a_\phi = 0$

$a_\phi \cdot a_\phi = 1$

① $$\int_a^b A \cdot dl = \int_a^b (\rho \cos\phi \, \hat{a}_\rho + \sin\phi \, \hat{a}_\phi) \cdot (\rho \, d\phi \, a_\phi)$$

$$\int_{60}^{30} \rho \sin\phi \, d\phi = \rho \int_{60}^{30} \sin\phi \, d\phi$$

$$\rho \left[-\cos\phi \right]_{60}^{30} = -2 \left[\cos 30 - \cos 60 \right]$$

$$= 1 - \sqrt{3}$$

② $\int_b^c A \cdot dl$ $\qquad \phi = 30 \qquad dl = d\rho \, \hat{a}_\rho$

$$\int (\rho\cos\phi\, a_\rho)\cdot(d\rho\, a_\rho) = \int \rho\cos\phi\, d\rho$$

$$\cos 30 \int_2^5 \rho\, d\rho = \cos 30 \left[\frac{\rho^2}{2}\right]_2^5$$

$$= \cos 30 \left[\frac{25}{2} - \frac{4}{2}\right] = \frac{21\sqrt{3}}{4}$$

③

$$\int_c^d A\cdot dl \qquad \rho = 5 \qquad dl = \rho\, d\phi\, \hat{a}_\phi$$

$$\int_c^d (\sin\phi\, a_\phi)\cdot(\rho\, d\phi\, a_\phi) = \int \rho\sin\phi\, d\phi$$

$$5\int_{30}^{60} \sin\phi\, d\phi = 5\left(-\cos\phi\Big]_{30}^{60}\right)$$

$$= -5\left[\cos 60 - \cos 30\right) = \frac{5}{3}(\sqrt{3} - 1)$$

④

$$\int_d^a A\cdot dl \qquad \phi = 60 \qquad dl = d\rho\, a_\rho$$

$$\int (\rho\cos\phi\, a_\phi)\cdot(d\rho\, a_\rho) = \cos 60 \int_5^2 \rho\, d\rho$$

$$= \cos 60 \left[\frac{\rho^2}{2}\right]_5^2 = \cos 60 \left[\frac{4}{2} - \frac{25}{2}\right] = -\frac{21}{4}$$

$$\oint_L A\cdot dl = \int^{①} + \int^{②} + \int^{③} + \int^{④} = 4.941$$

7- Conservative Fields

المجال المحافظ

A vector fields, $\vec{A}$, for which $\int_A^B \vec{A}\cdot d\vec{l}$ depends upon the path from A to B is called nonconservative. There are however, conservative fields for which $\int_A^B \vec{A}\cdot d\vec{l}$ is the same between the two given points regardless of what path we calculate it along.

$\vec{\nabla}\times\vec{A}=\vec{0}$ is a necessary and sufficient condition for $\int_A^B \vec{A}\cdot d\vec{l}$ to be independent of the path, that is, $\vec{\nabla}\times\vec{A}=\vec{0}$ for conservative fields and $\vec{\nabla}\times\vec{A}\neq\vec{0}$ for nonconservative fields.

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$\nabla\times\vec{A}=0$ محافظ

$\nabla\times\vec{A}\neq 0$ غير محافظ

components of $\vec{\nabla}\times\vec{A}$ are equal to zero.

$$\boxed{\text{If } \vec{A}=\vec{\nabla}f\text{, then } \vec{\nabla}\times\vec{A}=\vec{0}}$$

Conversely,

$$\boxed{\text{If } \vec{\nabla}\times\vec{A}=\vec{0}\text{, then we can find a function } f(x,y,z) \text{ for which } \vec{A}=\vec{\nabla}f}$$

Example: Show that $\vec{A} = (2xy - z^3)\vec{a}_x + x^2\vec{a}_y - (3xz^2 + 1)\vec{a}_z$ is conservative,

Solution:

$$\vec{\nabla} \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\vec{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{a}_z$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial(-3xz^2 - 1)}{\partial y} = 0\,,\ \frac{\partial A_y}{\partial z} = \frac{\partial(x^2)}{\partial z} = 0\,,\ \frac{\partial A_z}{\partial x} = \frac{\partial(-3xz^2 - 1)}{\partial x} = -3z^2$$

$$\frac{\partial A_x}{\partial z} = \frac{\partial(2xy - z^3)}{\partial z} = -3z^2\,,\ \frac{\partial A_y}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x\,,\ \frac{\partial A_x}{\partial y} = \frac{\partial(2xy - z^3)}{\partial z} = 2x$$

$$\vec{\nabla} \times \vec{A} = [0 - 0]\vec{a}_x - \left[-3z^2 + 3z^2\right]\vec{a}_y + [2x - 2x]\vec{a}_z = \vec{0} \text{ so } \vec{A} \text{ is conservative}$$

Example: Show that $\vec{A} = (2xy - z^3)\vec{a}_x + x^2\vec{a}_y - (3xz^2 + 1)\vec{a}_z$ is conservative,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^3 & x^2 & -3xz^2+1 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-3xz^2+1) - \frac{\partial}{\partial z}x^2\right]\hat{a}_x - \left[\frac{\partial}{\partial x}(-3xz^2+1) - \frac{\partial}{\partial z}(2xy - z^3)\right]\hat{a}_y$$

$$+ \left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy - z^3)\right]\hat{a}_z$$

$$= 0\,\hat{a}_x - [-3z^2 + 3z^2]\,a_y + [2x - 2x]\,\hat{a}_z$$

$$0\,\hat{a}_x - 0\,\hat{a}_y + 0\,a_z$$

$$= 0$$

$\vec{A}$ is conservative

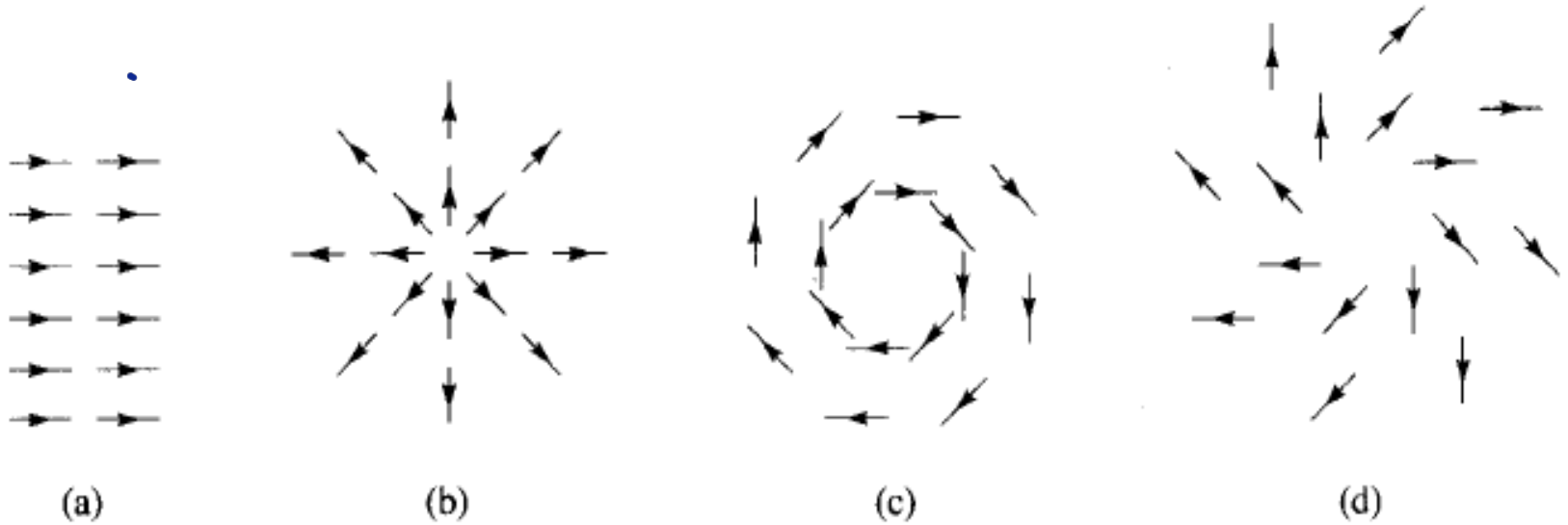
(a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$

(b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$

(c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$

(d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$

Figure 3.23 illustrates typical fields in these four categories.



Laplacian of a scalar $\nabla^2 V$

The **Laplacian** of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

$\nabla \cdot \nabla V$

Thus, in Cartesian coordinates,

Laplacian $V = \nabla \cdot \nabla V = \nabla^2 V$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

EXAMPLE 3.11

Find the Laplacian of the scalar fields of Example 3.3; that is,

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

a) $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$= \frac{\partial}{\partial x}\left(2\cos 2x\, e^{-z} \cosh y\right) + \frac{\partial}{\partial y}\left(e^{-z} \sin 2x \sinh y\right)$

$+ \frac{\partial}{\partial z}\left(-\sin 2x \cosh y\; e^{-z}\right)$

$= -4 \sin 2x\, e^{-z} \cosh y + e^{-z} \sin 2x \cosh y$

$+ \sin 2x \cosh y\, e^{-z}$

$\nabla^2 V = -2 \sin 2x \cosh y\, e^{-z}$

b) $U = \rho^2 z \cos 2\phi$

$\nabla U = \frac{\partial u}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial u}{\partial \phi} \hat{a}_\phi + \frac{\partial u}{\partial z} \hat{a}_z$

$= 2\rho z \cos 2\phi\, \hat{a}_\rho - 2\rho z \sin 2\phi\, a_\phi + \rho^2 \cos 2\phi\, \hat{a}_z$

$\nabla \cdot A = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho A_\rho + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$\nabla \cdot \nabla U = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(2\rho^2 z \cos 2\phi\right) + \frac{1}{\rho} \frac{\partial}{\partial \phi}\left(-2\rho z \sin 2\phi\right)$

$+ \frac{\partial}{\partial z}\left(\rho^2 \cos 2\phi\right)$

$\frac{1}{\cancel{\rho}}\left[4\cancel{\rho} z \cos 2\phi\right] - \frac{1}{\cancel{\rho}}\left[-4\cancel{\rho} z \cos 2\phi\right] + 0$

$\nabla^2 U = 4z \cos 2\phi + 4z \cos \phi$

EXAMPLE 3.10

For a vector field **A**, show explicitly that $\nabla \cdot \nabla \times \mathbf{A} = 0$; that is, the divergence of the curl of any vector field is zero.

Solution:

This vector identity along with the one in Practice Exercise 3.10 is very useful in EM. For simplicity, assume that **A** is in Cartesian coordinates.

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{A} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right), -\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right), \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{\partial^2 A_z}{\partial x\, \partial y} - \frac{\partial^2 A_y}{\partial x\, \partial z} - \frac{\partial^2 A_z}{\partial y\, \partial x} + \frac{\partial^2 A_x}{\partial y\, \partial z} + \frac{\partial^2 A_y}{\partial z\, \partial x} - \frac{\partial^2 A_x}{\partial z\, \partial y} \\
&= 0
\end{aligned}$$

PRACTICE EXERCISE 3.10

For a scalar field V, show that $\nabla \times \nabla V = 0$; that is, the curl of the gradient of any scalar field vanishes.

Answer: Proof.

Example

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times A = \left[\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right] a_x - \left[\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right] a_y + \left[\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right] a_z$$

$$\nabla \cdot P = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$$= \text{zero}$$

PRACTICE EXERCISE 3.10

For a scalar field V, show that $\nabla \times \underline{\nabla V} = 0$; that is, the curl of the gradient of any scalar field vanishes.

(A_x, A_y, A_z)

Answer: Proof.

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$$

$$\nabla \times \nabla V = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right]\hat{a}_x - \left[\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right]\hat{a}_y$$

$$+ \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right]$$

$$0\,\hat{a}_x + 0\,a_y + 0\,a_z$$

$$= 0$$