Homework 1

Fundamental Concepts (Vectors)

Question 1: Given the two vectors $\vec{A} = \hat{\imath} + \hat{\jmath}$ and $\vec{B} = \hat{\jmath} - \hat{k}$. Find

(a) $\vec{A} + \vec{B}$ and $|\vec{A} + \vec{B}|$

(b) $\vec{A} - \vec{B}$ and $|\vec{A} - \vec{B}|$

(d) $\vec{A} \times \vec{B}$ and $|\vec{A} \times \vec{B}|$

(e) $(\vec{A} + 2\vec{B}).(2\vec{A} - \vec{B})$

$\vec{A} = 1i + 1j + 0k$ $\quad \vec{B} = 0i + 1j - 1k$

a) $\vec{A}+\vec{B} = i + 2j - k$ $\quad |\vec{A}+\vec{B}| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$

b) $\vec{A}-\vec{B} = i + k$ $\quad |\vec{A}-\vec{B} = \sqrt{1^2+1^2} = \sqrt{2}$

d) $\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} k$

$(-1-0)i - (-1-0)j + (1-0)k$

$-1i + 1j + 1k$

$A \times B = -i + j + k$

$|A \times B| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{3}$

e) $(A+2B) \cdot (2A-B)$

$\vec{A} = 1i + 1j + 0k$

$\vec{B} = 0i + 1j - 1k$

$2\vec{A} = 2i + 2j + 0k$

$2\vec{B} = 0i + 2j - 2k$

$(1i + 1j + 0k + 0i + 2j - 2k) \cdot (2i + 2j + 0k - 0i - 1j + k)$

$(i + 3j - 2k) \cdot (2i + j + k)$

$(1 \times 2) + (3 \times 1) + (-2 \times 1)$

$2 + 3 - 2 = 3$

Question 2: Given $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{B} = \hat{\imath} + 2\hat{\jmath}$, $\vec{C} = 2\hat{\jmath} - \hat{k}$. Find

(a) $\vec{A} + \vec{B} - \vec{C}$

(b) $\vec{A}.(\vec{B} + \vec{C})$ and $(\vec{A} + \vec{B}).\vec{C}$

(c) $\vec{A}.(\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}).\vec{C}$

(d) $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \times \vec{C}$

a) $A+B-C = i+j+k + i+2j - 2j+k$

$= 2i + j + 2k$

b) $A.(B+C) = (i+j+k) \cdot (i+4j-k)$

$(1\times1) + (1\times4) + (1\times-1)$

$1 + 4 - 1 = 4$

$(A+B).C = (i+j+k+i+2j) \cdot (2j-k)$

$(2i + 3j + k) \cdot (2j - k) = (3\times2) + (1\times-1) = 5$

c) $A.(B\times C) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix}$

$= \begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} 1 - \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} 1 + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} 1$

$= (-2-0) - (-1-0) + (2-0) =$

$-2 + 1 + 2 = 1$

$(A\times B).C$ same answer

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$a.c = 2-1 = 1$

$a.b = 1+2 = 3$

$b(a.c) - c(a.b)$

$b(1) - c(3)$

$i + 2j - 6j + 3k$

d) $A\times(B\times C)$

$B\times C = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -2i + j + 2k$

$$A \times (B \times C) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{vmatrix} = i - 4j + 3k$$

$$= i - 4j + 3k$$

$$(A \times B) \times C =$$

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2i + j + k$$

$$(A \times B) \times C = \begin{vmatrix} i & j & k \\ -2 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -3i - 2j - 4k$$

Question 3: Find the angle between the vectors $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{B} = \hat{\imath} + \hat{\jmath}$.

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|A||B|} = \frac{2}{\sqrt{3}\sqrt{2}}$$

$$\vec{A} \cdot \vec{B} = (1 \times 1) + (1 \times 1) + (1 \times 0) = 2$$

$$|\vec{A}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\vec{B}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \cos^{-1}\left[\frac{2}{\sqrt{3}\sqrt{2}}\right] = 35.26^\circ$$

Question 4: Given the time-varying vectors $\vec{A} = \cos\omega t\ \hat{\imath} + \sin\omega t\ \hat{\jmath}$ and $\vec{B} = t\,\hat{\imath} + t^2\,\hat{\jmath} + t^3\,\hat{k}$. Find

(a) $d\vec{A}/dt$ and $|d\vec{A}/dt|$ (b) $d^2\vec{B}/dt^2$ and $|d^2\vec{B}/dt^2|$

(c) $d(\vec{A}.\vec{B})/dt$ (d) $d(\vec{A}\times\vec{B})/dt$

a) $\frac{d\vec{A}}{dt} = (-\omega \sin\omega t)i + (\omega\cos\omega t)j$

$$\left|\frac{d\vec{A}}{dt}\right| = \sqrt{\omega^2\sin^2\omega t + \omega^2\cos^2\omega t}$$

$$= \sqrt{\omega^2}\sqrt{\sin^2\omega t + \cos^2\omega t}$$

$$= \omega$$

b) $\vec{B} = ti + t^2 j + t^3 k$

$$\frac{d\vec{B}}{dt} = i + 2tj + 3t^2 k$$

$$\frac{d^2\vec{B}}{dt^2} = 0i + 2j + 6tk = 2j + 6tk$$

$$\left|\frac{d^2\vec{B}}{dt^2}\right| = \sqrt{2^2 + 6^2t^2} = \sqrt{4 + 36t}$$

c) $\frac{d(A.B)}{dt} = \frac{d}{dt}(t\cos\omega t + t^2\sin\omega t)$

$= -t\omega\sin\omega t + \cos\omega t + t^2\omega\cos\omega t + 2t\sin\omega t$ ✓

$= t(2-\omega)\sin\omega t + (1 + t^2\omega)\cos\omega t$

d) $\frac{d(A \wedge B)}{dt} \Rightarrow A \times B = \begin{vmatrix} i & j & k \\ \cos\omega t & \sin\omega t & 0 \\ t & t^2 & e^3 \end{vmatrix}$

$$= \begin{vmatrix} \sin\omega t & 0 \\ t^2 & e^3 \end{vmatrix} i - \begin{vmatrix} \cos\omega t & 0 \\ t & t^3 \end{vmatrix} j + \begin{vmatrix} \cos\omega t & \sin\omega t \\ t & t^2 \end{vmatrix} k$$

$$\vec{A} \times \vec{B} = (t^3 \sin\omega t)\, i - t^3 \cos\omega t\, j + (t^2 \cos\omega t - t \sin\omega t)\, k$$

$$= (3t^2 \sin\omega t + t^3 \omega \cos\omega t)\, i - (3t^2 \cos\omega t - t^3 \omega \sin\omega t)\, j$$

$$(2t \cos\omega t - t^2 \omega \sin\omega t - \sin\omega t - t\omega\cos\omega t)\, k$$

Question 5: For what values of q are the two vectors $\vec{A} = \hat{\imath} + \hat{\jmath} + q\,\hat{k}$ and $\vec{B} = q\,\hat{\imath} - 2\hat{\jmath} + 2q\,\hat{k}$ perpendicular to each other?

حتى يكون المتجهين متعامدين يجب ان يكون $\vec{A} \cdot \vec{B} = 0$

$\vec{A} = i + j + qk$ $\qquad$ $\vec{B} = qi - 2j + 2qk$

$\vec{A} \cdot \vec{B} = (1 \times q) + (1 \times 2) + (q \times 2q) = 0$

$= q - 2 + 2q^2 = 0$

$2q^2 + q - 2 = 0$

$q = -1.28 \qquad q = 0.78$